

Mesoscopic photon heat transistor

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We show that the heat transport between two bodies, mediated by electromagnetic fluctuations, can be controlled with an intermediate quantum circuit - leading to the device concept Mesoscopic Photon Heat Transistor (MPHT). Our theoretical analysis is based on a novel Meir-Wingreen-Landauer type of conductance formula, which gives the photonic heat current through an arbitrary circuit element coupled to two dissipative reservoirs at finite temperatures. As an illustration we present an exact solution for the case when the intermediate circuit can be described as an electromagnetic resonator. We discuss in detail how the MPHT can be implemented experimentally in terms of a flux-controlled SQUID circuit.

Problems involving interactions between quantum particles and electromagnetic fields have a long and rich history. During the last decades developments in mesoscopic physics have provided unprecedented possibilities to engineer the electron-photon interactions. A major advantage in mesoscopic systems is their versatility which allows a high-level of control in their design and operation. Recent advances in this field include impressive quantum state manipulations involving single photons in circuit cavity QED experiments [1] and a demonstration of single-channel photon heat transport [2].

At low temperatures, when phonon modes become effectively frozen, photonic heat conduction becomes the dominant channel for thermal transport [2, 3]. In this Letter we study photonic heat transport in a structure consisting of two reservoirs coupled via an intermediate electric circuit. The reservoirs are assumed to behave as linear dissipative circuit elements and are thereby fully characterized by their response functions and temperatures. Furthermore, at low temperatures the wavelengths of relevant field fluctuations are much longer than a typical system size so the reservoirs can be effectively considered as lumped elements. Within these assumptions we apply the Caldeira-Leggett procedure and model the reservoirs as continuous distributions of harmonic oscillators. By applying nonequilibrium Green's function methods we derive a formally exact Meir-Wingreen-Landauer-type formula [4] for the heat current through the structure. Our formula involves the *noise power* of the intermediate circuit, in the presence of the coupling to the leads, and serves as a general starting point for solving the heat transport problem. We find an exact solution for the heat current flowing through an electromagnetic resonator circuit and show explicitly that, in analogy with the transistor effect in charge transport problems, the heat flow through the structure can be modulated by applying an external control to the middle circuit. We suggest that an experimental demonstration of the

heat-transistor action can be achieved by using a Superconducting QUantum Interference Device (SQUID) circuit as the tunable resonator. Thus, the magnetic flux controlled heat current is a photonic analogue to the gate voltage controlled electronic heat current recently demonstrated in Ref. [5].

Now we turn to the technical derivation of the general formula for the heat current in the system consisting of a left and a right reservoir, and an arbitrary quantum circuit between them (Fig. 1). We treat the problem by employing a nonequilibrium Green's function method analogous to those used earlier in electron and heat transport problems [4, 6, 7, 8, 9]. The reservoirs are described by quadratic boson fields and, according to the Caldeira-Leggett prescription, can be thought of as arbitrary linear electric circuits by choosing specific distributions of frequencies ω_j and couplings g_j (introduced below) [10, 11]. The total Hamiltonian is assumed to be of the form $H = H_L + H_R + H_M + H_C$, where

$$H_{L/R} = \sum_{j \in L/R} \hbar \omega_j (\hat{a}_j^\dagger \hat{a}_j + 1/2), \quad (1)$$

and the inductive coupling term is

$$H_C = M \hat{I} (\hat{i}_L + \hat{i}_R), \quad (2)$$

which involves the current operators for the central region \hat{I} and for the reservoirs $\hat{i}_{L/R} = \sum_{j \in L/R} g_j (\hat{a}_j + \hat{a}_j^\dagger)$, respectively. The specific form of the Hamiltonian and the current operator of the middle region do not need to be specified at this point (below we shall treat a specific example). The mutual inductances between the middle circuit and the leads are assumed to be equal, though this is not necessary in the following derivation. A capacitive coupling between the reservoirs and the middle system can be treated in close analogy with the inductive coupling studied here [12].

The reservoir Hamiltonians $H_{L/R}$ do not commute with the total Hamiltonian H , thus giving a rise to an

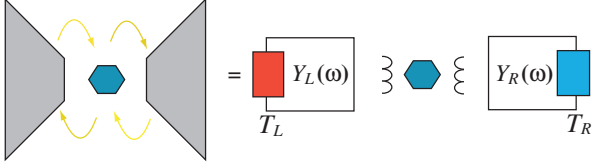


FIG. 1: Temperature gradient between the left and right reservoirs induces photonic heat current through an arbitrary quantum circuit coupled to them. Heat flows from left to right when $T_L > T_R$. The reservoirs are assumed to behave as linear dissipative elements and they couple only through the middle circuit. In principle a direct coupling between the reservoirs always exists but in practice it can be made negligible by an appropriate sample design.

energy flow in the structure. This energy flow is characterized by a heat current $J_{L/R}$ defined as

$$\begin{aligned} J_{L/R}(t) &= \langle \dot{H}_{L/R} \rangle = iM \sum_{j \in L/R} \left[g_j \omega_j \langle \hat{a}_j(t) \hat{I}(t) \rangle - \text{h.c.} \right] \\ &= -2M \text{Re} \sum_{j \in L/R} g_j \omega_j G_j^<(t, t), \end{aligned} \quad (3)$$

where $G_j^<(t, t') \equiv -i \langle \hat{a}_j(t') \hat{I}(t) \rangle$. The transport problem is reduced to finding the lesser Green's function $G_j^<(t, t')$ which can be derived by the equation-of-motion technique [6, 13]. Following the standard prescription [6], we first consider the equilibrium zero-temperature time-ordered correlation functions, which obey

$$(i\partial_{t'} - \omega_j) \langle T[\hat{a}_j(t') \hat{I}(t)] \rangle = \frac{Mg_j}{\hbar} \langle T[\hat{I}(t') \hat{I}(t)] \rangle, \quad (4)$$

or, after a formal integration,

$$\langle T[\hat{a}_j(t') \hat{I}(t)] \rangle = \frac{Mg_j}{\hbar} \int dt_1 \langle T[\hat{I}(t) \hat{I}(t_1)] \rangle D_j(t_1 - t'), \quad (5)$$

where $D_j(t_1 - t')$ is the free reservoir Green's function. In nonequilibrium, this equation holds on the Keldysh contour, and using the analytical continuation rules known as Langreth's theorem [6], we obtain

$$\begin{aligned} G_j^<(t, t') &= \frac{Mg_j}{\hbar} \int dt_1 \left[\langle \hat{I}(t) \hat{I}(t_1) \rangle^r D_j^<(t_1 - t') + \right. \\ &\quad \left. + \langle \hat{I}(t) \hat{I}(t_1) \rangle^< D_j^a(t_1 - t') \right], \end{aligned} \quad (6)$$

where the superscripts r , a and $<$ stand for "retarded", "advanced" and "lesser", respectively. Explicitly, the current correlation functions are $\langle \hat{I}(t) \hat{I}(t') \rangle^r = -i\theta(t - t') \langle [\hat{I}(t), \hat{I}(t')] \rangle$ and $\langle \hat{I}(t) \hat{I}(t') \rangle^< = -i \langle \hat{I}(t') \hat{I}(t) \rangle$. In a steady state $G_j^<(t, t') = G_j^<(t - t')$, and it is convenient to introduce the Fourier transform:

$$G_j^<(\omega) = \frac{Mg_j}{\hbar} \left[\langle \hat{I} \hat{I} \rangle^r(\omega) D_j^<(\omega) + \langle \hat{I} \hat{I} \rangle^<(\omega) D_j^a(\omega) \right], \quad (7)$$

where $D_j^a(\omega) = 1/(\omega - \omega_j - i\eta)$, $D_j^<(\omega) = -i2\pi n(\omega_j) \delta(\omega - \omega_j)$, and $n(\omega)$ is the Bose function. We thus obtain

$$\begin{aligned} J_L &= 2 \sum_{j \in L} \frac{M^2 g_j^2 \omega_j}{2\pi\hbar} \int_{-\infty}^{\infty} d\omega \left[-\text{Im} \langle \hat{I} \hat{I} \rangle^r(\omega) 2\pi n(\omega_j) \delta(\omega - \omega_j) \right. \\ &\quad \left. + \text{Im} \langle \hat{I} \hat{I} \rangle^<(\omega) \pi \delta(\omega - \omega_j) \right]. \end{aligned} \quad (8)$$

To make a connection to quantities with a clear physical interpretation, it is useful to express Eq. (8) in terms of the noise power. The noise power for an observable \hat{A} is defined as $S_A(\omega) = \int_{-\infty}^{\infty} dt e^{i\omega(t-t')} \langle \hat{A}(t) \hat{A}(t') \rangle$, so the current noise in the left lead is

$$\begin{aligned} S_{i_L}(\omega) &= \sum_{j \in L} g_j^2 [n(\omega_j) 2\pi \delta(\omega + \omega_j) + \\ &\quad + (n(\omega_j) + 1) 2\pi \delta(\omega - \omega_j)]. \end{aligned} \quad (9)$$

In terms of the noise power Eq. (8) becomes

$$\begin{aligned} J_L &= 2M^2 \int_0^{\infty} \frac{d\omega\omega}{2\pi\hbar} \left[-\text{Im} \langle \hat{I} \hat{I} \rangle^r(\omega) S_{i_L}(-\omega) \right. \\ &\quad \left. + \text{Im} \langle \hat{I} \hat{I} \rangle^<(\omega) \frac{1}{2} (S_{i_L}(\omega) - S_{i_L}(-\omega)) \right]. \end{aligned} \quad (10)$$

Also the \hat{I} -correlation functions can be written in terms of noise power: $-\text{Im} \langle \hat{I} \hat{I} \rangle^r(\omega) = \frac{1}{2} (S_I(\omega) - S_I(-\omega))$ and $\text{Im} \langle \hat{I} \hat{I} \rangle^<(\omega) = -S_I(-\omega)$, which yields

$$J_L = M^2 \int_0^{\infty} \frac{d\omega\omega}{2\pi\hbar} [S_I(\omega) S_{i_L}(-\omega) - S_I(-\omega) S_{i_L}(\omega)]. \quad (11)$$

A similar expression holds for J_R ; in steady-state situations $J \equiv J_L = -J_R$. Formulas (10) and (11) contain the free reservoir noise functions which can be obtained straightforwardly from the admittances $Y_{L/R}(\omega)$ and temperatures $T_{L/R}$ by applying the Fluctuation-Dissipation theorem [10]:

$$S_{i_{L/R}}(\omega) = \text{Re}[Y_{L/R}(\omega)] \hbar\omega [\coth(\beta_{L/R} \hbar\omega/2) + 1]. \quad (12)$$

With the help of Eq. (12), the heat current (11) can be written as

$$\begin{aligned} J_L &= \int_0^{\infty} \frac{d\omega\omega^2 M^2}{2\pi} \{ 2 [S_I(\omega) - S_I(-\omega)] \text{Re}[Y_L(\omega)] n_L(\omega) \\ &\quad - S_I(-\omega) 2 \text{Re}[Y_L(\omega)] \} \end{aligned} \quad (13)$$

If the lead admittances share the same frequency dependence, $Y_L(\omega) = cY_R(\omega)$ with some constant c , the stationary heat current can be expressed as $J = J_L/(c + 1) - cJ_R/(c + 1)$ and cast into the Landauer form:

$$\begin{aligned} J &= M^2 \int_0^{\infty} \frac{d\omega\omega^2}{2\pi} [S_I(\omega) - S_I(-\omega)] \\ &\quad \times \frac{2 \text{Re}[Y_L(\omega)] \text{Re}[Y_R(\omega)]}{\text{Re}[Y_L(\omega)] + \text{Re}[Y_R(\omega)]} [n_L(\omega) - n_R(\omega)]. \end{aligned} \quad (14)$$

Equation (14) is the main formal result of this Letter. We emphasize the strong analogue between this result and the Meir-Wingreen conductance formula: here the “bosonic thermal window” ($n_L - n_R$) replaces the “fermionic voltage window” ($n_L^F - n_R^F$), the real part of the admittance plays the role of the line-width function $\Gamma_{L/R}$, and the noise power $S_I(\omega) - S_I(-\omega)$ of the central region, which needs to be evaluated separately, replaces the central region spectral function. Since S_I contains information of the internal dynamics of the middle region *in the presence of coupling to the reservoirs*, its evaluation may be difficult indeed, and only in special cases analytic progress can be expected. In the following we calculate $S_I(\omega)$ for an electromagnetic resonator and illustrate how the reservoir admittances come into play through the self-energy of the middle circuit.

Suppose now that the mediating quantum circuit is an electromagnetic resonator with inductance L and capacitance C (see Fig. 2 a). The central region Hamiltonian then takes the form $H_M = \hbar\omega_0(\hat{b}^\dagger\hat{b} + \frac{1}{2})$, and the current operator is $\hat{I} = I_0(\hat{b} + \hat{b}^\dagger)$, where $I_0 = \sqrt{\hbar\omega_0/2L}$ and $\omega_0 = 1/\sqrt{LC}$; \hat{b} , \hat{b}^\dagger are bosonic creation and annihilation operators, $[\hat{b}, \hat{b}^\dagger] = 1$. To evaluate (14) one needs to find the retarded function $\langle \hat{I}(t)\hat{I}(t') \rangle^r = -i\theta(t-t')\langle [\hat{I}(t), \hat{I}(t')] \rangle$ which can be expressed as sum of four different retarded functions

$$\langle \hat{I}\hat{I} \rangle^r(\omega) = I_0^2 \left[\langle \hat{b}\hat{b}^\dagger \rangle^r(\omega) + \langle \hat{b}^\dagger\hat{b} \rangle^r(\omega) + \langle \hat{b}\hat{b} \rangle^r(\omega) + \langle \hat{b}^\dagger\hat{b}^\dagger \rangle^r(\omega) \right]. \quad (15)$$

Note the presence of the anomalous functions $\langle \hat{b}(t)\hat{b}(t') \rangle^r$ and $\langle \hat{b}^\dagger(t)\hat{b}^\dagger(t') \rangle^r$; they are important and come into play because the interaction term also includes the non-rotating-wave terms $\hat{a}_j\hat{b}$ and $\hat{a}_j^\dagger\hat{b}^\dagger$. By using an equation-of-motion technique [6, 13] we find a closed set of equations

$$\begin{aligned} (\omega - \omega_0 + i\eta)\langle \hat{b}\hat{b}^\dagger \rangle^r &= 1 + \Sigma^r(\omega) \left[\langle \hat{b}\hat{b}^\dagger \rangle^r + \langle \hat{b}^\dagger\hat{b} \rangle^r \right] \\ (\omega + \omega_0 + i\eta)\langle \hat{b}^\dagger\hat{b} \rangle^r &= -\Sigma^r(\omega) \left[\langle \hat{b}\hat{b}^\dagger \rangle^r + \langle \hat{b}^\dagger\hat{b} \rangle^r \right] \\ (\omega + \omega_0 + i\eta)\langle \hat{b}^\dagger\hat{b} \rangle^r &= -1 - \Sigma^r(\omega) \left[\langle \hat{b}\hat{b} \rangle^r + \langle \hat{b}^\dagger\hat{b}^\dagger \rangle^r \right] \\ (\omega - \omega_0 + i\eta)\langle \hat{b}\hat{b} \rangle^r &= \Sigma^r(\omega) \left[\langle \hat{b}\hat{b} \rangle^r + \langle \hat{b}^\dagger\hat{b}^\dagger \rangle^r \right], \end{aligned} \quad (16)$$

where the retarded self-energy is given by

$$\Sigma^r(\omega) = \frac{(MI_0)^2}{\hbar^2} \sum_{j \in L, R} g_j^2 \left(\frac{1}{\omega - \omega_j + i\eta} - \frac{1}{\omega + \omega_j + i\eta} \right). \quad (17)$$

The reservoir admittances are given by the Kubo formula

$$\begin{aligned} Y_L(\omega) &= \frac{i\langle \hat{i}_L \hat{i}_L \rangle^r(\omega)}{\hbar\omega} \\ &= \frac{i}{\hbar\omega} \sum_{j \in L} g_j^2 \left(\frac{1}{\omega - \omega_j + i\eta} - \frac{1}{\omega + \omega_j + i\eta} \right), \end{aligned} \quad (18)$$

so the self-energy is related to the reservoir admittances through relation

$$\Sigma^r(\omega) = -\frac{iM^2I_0^2\omega}{\hbar} [Y_L(\omega) + Y_R(\omega)]. \quad (19)$$

The algebraic system (16) is readily solved yielding

$$\langle \hat{I}\hat{I} \rangle^r(\omega) = \frac{-i\hbar\omega}{\omega F(\omega + i\eta) + M^2\omega^2(Y_L(\omega) + Y_R(\omega))}, \quad (20)$$

where we have introduced shorthand $F(\omega) \equiv (i\hbar/I_0^2)(\omega^2 - \omega_0^2)/(2\omega_0)$. By extracting the imaginary part of Eq. (20) and recalling the relation $-2\text{Im}\langle \hat{I}\hat{I} \rangle^r(\omega) = S_I(\omega) - S_I(-\omega)$ we find

$$\begin{aligned} J &= \int_0^\infty \frac{d\omega}{2\pi} \frac{4\hbar\omega^5 M^4 \text{Re}[Y_L(\omega)] \text{Re}[Y_R(\omega)]}{|\omega F(\omega) - M^2\omega^2(Y_L(\omega) + Y_R(\omega))|^2} \times \\ &\quad \times [n_L(\omega) - n_R(\omega)]. \end{aligned} \quad (21)$$

The result (21) is valid for arbitrary reservoirs admittances, as long as the intermediate circuit can be described as an oscillator. It can be shown that the upper limit of heat current (21) is given by the universal single-channel heat conductance quantum $G_Q = \pi^2 k_B^2 T/3\hbar$, [2, 3, 9, 14, 15, 16], an exact parallel with the electrical single-channel conductance $G_0 = 2e^2/\hbar$, which follows from the Meir-Wingreen formula for a noninteracting resonant level model at resonance [6].

In the following we assume that the reservoirs are identical and consist of a resistor, a capacitor and an inductor in series. Thus the admittances can be written as $Y_L(\omega) = Y_R(\omega) = \frac{R^{-1}}{1 + iQ(\omega/\omega_R - \omega_R/\omega)}$, where R , Q and ω_R are an effective resistance, a quality factor and a resonance frequency of the reservoir. The net heat flow is determined by five dimensionless parameters $M^2\omega_0/LR$, ω_0/ω_R , Q , $\hbar\omega_0/k_B T_L$ and $\hbar\omega_0/k_B T_R$ where T_L and T_R are temperatures of the reservoirs. By externally controlling the parameters of the middle circuit it is possible to tune the heat flow through the structure. A practical way to realize this is to employ a dc-SQUID as a middle circuit and apply a magnetic flux Φ through it (see Fig. 2 b)). At low temperatures the system can be modeled by an LC-oscillator with a tunable inductance $L(\Phi) \approx L/|\cos(\pi\Phi/\Phi_0)|$ where $\Phi_0 = h/2e$ is the flux quantum and L is determined by the Ambegaokar-Baratoff critical current [17]. The LC-oscillator description of the SQUID circuit is expected to be accurate within realistic parameter values at the experimentally

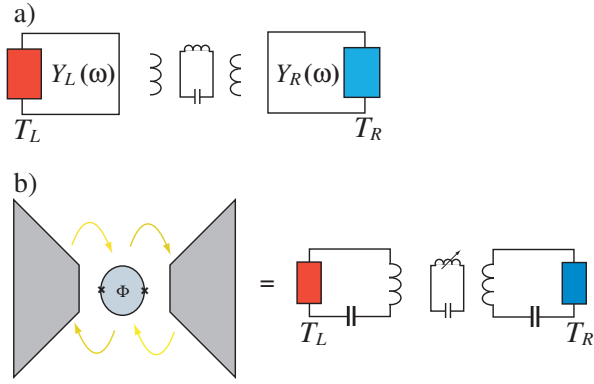


FIG. 2: An electromagnetic resonator as the intermediate circuit (a)). In b) the mediate resonator circuit with a tunable inductance is realized by a dc-SQUID with an external magnetic flux bias Φ . The external flux can be used to modulate the heat current through the structure and plays an analogous role to the gate voltage in electronic transistor.

verified crossover temperature $T_{cr} \sim 100$ mK below which the photonic thermal conductance should dominate [2]. A numerical evaluation of Eq. (21) is presented in Fig. 3, which shows the heat flow as a function of the external bias flux Φ . The maximum flow is obtained at integer values of Φ/Φ_0 , whereas at half-integer values the reservoirs are thermally decoupled. At low quality factors the reservoirs are more efficiently matched and the system has a better thermal coupling. By increasing the coupling parameter $M^2\omega_0/LR$ the maximum value of the heat flow could be enhanced closer to the single-channel maximum value.

In summary, we have studied photon heat transport in nanoelectronic circuits based on a novel Meir-Wingreen-Landauer formula in a two-terminal geometry. This formula expresses the heat current in terms of the admittances of the heat reservoirs, and the noise power of the intermediated mesoscopic circuit. The formula can serve as a starting point for an analysis of photon heat transport in a wide range of applications. As an example, we present an exact solution to the transport problem in the case of an electromagnetic resonator playing the role of the mediating circuit. We propose a new device concept, a mesoscopic photon heat transistor, where the heat current through the structure can be strongly modulated by the external magnetic flux through a dc-SQUID loop.

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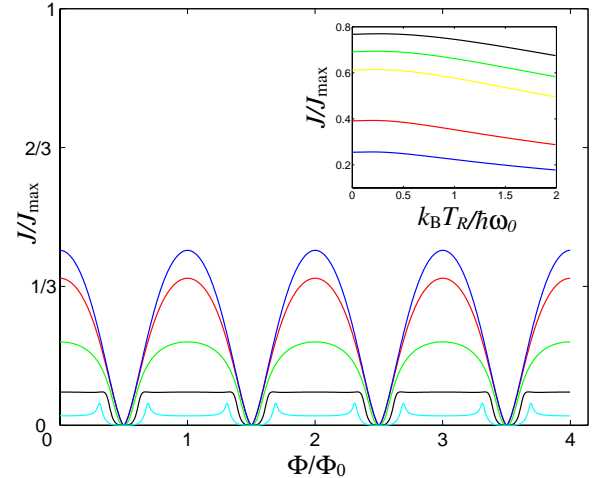


FIG. 3: Heat flow through the SQUID structure as a function of applied flux Φ , corresponding to parameters $M^2\omega_0/LR = 1$, $\omega_0/\omega_R = 1$ and $T_R = T_L/2 = \hbar\omega_0$. The different curves correspond to different reservoir Q values, $Q = 0$ (blue), $Q = 0.1$ (red), $Q = 0.5$ (green), $Q = 2$ (black) and $Q = 10$ (cyan). The heat flux is normalized with respect to the universal single-channel maximum value $J_{\max} = \frac{\pi^2 k_B^2}{6\hbar} (T_L^2 - T_R^2)$. In the inset the heat flux is plotted as a function of the temperature of the right reservoir corresponding to the parameters $\omega_0/\omega_R = 1$, $T_L = 2\hbar\omega_0$, $Q = 0.1$ and $\Phi = 0 \bmod \Phi_0$. The different curves correspond to $M^2\omega_0/LR = 0.5$ (blue), $M^2\omega_0/LR = 1$ (red), $M^2\omega_0/LR = 3$ (yellow), $M^2\omega_0/LR = 5$ (green) and $M^2\omega_0/LR = 10$ (black).